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ON CONVERTING SORTIES KILLED TO AIRCRAFT KILLED IN COMBAT MODELS THAT USE ATTRITION EQUATIONS

Eleanor L. Schwartz

September 1988

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Some combat simulation models that use attrition equations employ a two-stage procedure to compute attrition to aircraft: first, a number of sorties killed is computed and then it is converted to a number of aircraft killed. This paper uses probability theory to derive several different formulas that could be used to convert sorties killed to aircraft killed in the case where sortie rates are greater than 1. (If sortie rates do not exceed 1, it is reasonable to let aircraft killed equal sorties killed.) Several possible formulas for the expected number of successful sorties flown also are derived. A number of inequality relationships among these formulas are proved. The commonly used formula: aircraft killed = sorties killed ÷ sortie rate, is found to produce the smallest results of all the formulas derived. One of the alternative formulas has been implemented in the NAVMOD naval combat model.

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INSTITUTE FOR DEFENSE ANALYSES

IDA Independent Research Program

PREFACE

This paper was prepared under the Institute for Defense Analyses' Central Research Program. It is an expansion and motivation of some concepts incorporated into IDA's NAVMOD model of naval combat (Reference [4]).

The author is grateful to Dr. Lowell Bruce Anderson for many comments that were useful in the preparation of this paper, and to Dr. Peter Brooks for his thorough review. The author also wishes to thank Dr. Royce Kneece, OSD, the original developer of one of the alternative formulas for converting sorties killed to aircraft killed; his work has motivated this paper.

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ABSTRACT

Some combat simulation models that use attrition equations employ a two-stage procedure to compute attrition to aircraft: first, a number of sorties killed is computed and then it is converted to a number of aircraft killed. This paper uses probability theory to derive several different formulas that could be used to convert sorties killed to aircraft killed in the case where sortie rates are greater than 1. (If sortie rates do not exceed 1, it is reasonable to let aircraft killed equal sorties killed.) Several possible formulas for the expected number of successful sorties flown also are derived. A number of inequality relationships among these formulas are proved. The commonly used formula: aircraft killed = sorties killed ÷ sortie rate, is found to produce the smallest results of all the formulas derived. One of the alternative formulas has been implemented in the NAVMOD naval combat model.

CONTENTS

FIGURE

A. INTRODUCTION

In general, an attrition equation takes a number of shooters, S, and a number of targets, T, and computes a number of targets killed

$$\dot{T} = f(S,T;\underline{p}) , \qquad (1)$$

where $\underline{p}$ is some set of effectiveness parameters (e.g., detection and kill probabilities). Suppose, however, that the targets (and perhaps the shooters too) are aircraft with some sortie rate. Let r_t be the sortie rate for targets and r_s, the rate for shooters. One way of computing the number of targets killed is to first compute the number of target sorties killed

$$\dot{T}_s = f(r_s S, r_t T;\underline{p}) \qquad (2)$$

and to then let

$$\dot{T} = \begin{cases} \dot{T}_s/r_t & \text{if } r_t > 1 \\ \dot{T}_s & \text{if } r_t \leq 1 . \end{cases} \qquad (3)$$

If sortie rates do not exceed 1.0, they can be regarded as "availability fractions," and the above attrition method is certainly reasonable. In this case, the attrition method treats each sortie as an aircraft participating in combat, thus a sortie killed results in an aircraft killed.

If the sortie rate r_t is greater than 1.0, the above approach results in the percentage of sorties killed equaling the percentage of aircraft killed. Potential problems can arise, however, because of the dynamics of flying sorties over a time period of combat. In the words of L.B. Anderson (Reference [3]):

> This method for considering sortie rates other than 1.0 is the same as is used in IDAGAM I (see References [1] and [2]) and appears to give reasonable results. However, it cannot be theoretically justified for sortie rates greater than 1.0, because OPTSA (and IDAGAM I) assesses attrition once per day, while multiple sorties per day imply that attrition can occur on the first sortie (which would affect the outcome one way) or on later sorties (which would affect the outcome a different way). If further research indicates that this variance in outcome is significant, then attrition should be assessed more frequently than once per day. For the time being, note that if the number of sorties killed as given by [equation (2)] is correct, then (for $r_t > 1$) [equation (3)] would give a lower bound on the number of

aircraft killed--because, if less than $\dot{T}_s/r_t$ aircraft are killed, then the number of sorties killed would be less than $\dot{T}_s$, even if all aircraft are killed on their first sortie. On the other hand, for $r_t > 1$, [equation (2)] might overestimate the number of sorties killed--because some aircraft on both sides would be killed on their first sortie; and so there would be, on the average, less than r_tT targets and less than r_sS shooters.

Suppose that it is not desired (perhaps because of computer time limitations) to assess attrition more than once per time period (i.e., assessment cycle) and sortie rates (per time period) exceed 1.0. Then the results of a combat simulation could be sensitive to whether or not aircraft kills are determined by first computing sorties killed using an equation like (2) and, if so, to the particular equation used to compute aircraft killed from sorties killed. It might thus be instructive to develop and examine several approaches for computing aircraft killed when sortie rates exceed 1.0. As a start, this paper addresses the more restrictive question: given that it is reasonable to compute sorties killed using an attrition equation such as (2), what are some possible alternatives to equation (3) for converting sorties killed to aircraft killed when sortie rates exceed 1.0?

This paper is partially motivated by the fact that one such alternative formula was developed earlier and has been implemented as an option (when sortie rates exceed 1.0) in some places of the NAVMOD model (Reference [4]). This formula,

$$\dot{T} = T\left[1 - \left(1 - \frac{\dot{T}_s}{r_tT}\right)^{r_t}\right], \tag{4}$$

has the interpretation that the probability an aircraft is killed on a particular sortie can be approximated by the total sorties killed divided by the total sorties, i.e., $\dot{T}_s/r_tT$; an aircraft is killed unless it survives all of the r_t sorties it flies. The natural questions arise: How does equation (4) compare to equation (3)? What are some other interpretations of equation (4)? Are there yet other equations that might provide reasonable ways of converting sorties killed to aircraft killed?

This paper explores some aspects of these issues. Section B derives some results on sortie dynamics in the simplest attrition process: the one where a sortie is killed with some constant probability. Sections C and D derive and explore several formulas that might be reasonable to convert sorties killed to aircraft killed. The results are presented in Section C; proofs are given in Section D. As an excursion, Section E briefly examines a certain probabilistic experiment, and draws some connections between it and the results of Sections C and D. Section F summarizes the paper's results and suggests some further research topics.

The formulas for aircraft killed developed in this paper all assume that sortie rates are greater than or equal to 1.0, and do not make sense for sortie rates less than 1.0. In this latter case, however, it is perfectly reasonable to let aircraft killed equal sorties killed. Furthermore, all the formulas have the "continuity" property that when the formulas are evaluated at a sortie rate of (exactly) 1.0, the resultant number of aircraft killed is equal to the number of sorties killed. Accordingly, throughout the remainder of this paper, all sortie rates are assumed to be greater than or equal to 1.0.

B. EXPECTED AIRCRAFT KILLS AND SUCCESSFUL SORTIES IN A SIMPLE ATTRITION PROCESS

This section uses a different notation than the preceding section. Suppose that there are A aircraft. An aircraft flying a sortie is killed on that sortie with probability p (regardless of the number of aircraft or the number of enemy attackers). If killed, an aircraft (obviously) can fly no more sorties. If not killed, the aircraft will fly another sortie, unless it has already flown R successful sorties (where a "successful" sortie is simply a sortie on which the aircraft has not been killed). R represents a sortie rate; assume that both A and R are positive integers.

The expected number of aircraft killed clearly is given by

$$\dot{A} = A\,[1 - (1-p)^R]\;;$$

note the similarity of this equation to equation (4) of the previous section. A more interesting quantity is the expected number of successful sorties flown (which equals the "total potential number of sorties," AR, less the number of sorties "killed or forestalled"). This is given by the following

<u>Proposition</u>: In the attrition process described above, the expected number of successful sorties flown is given by

$$\frac{A(1-p)}{p}\,[1 - (1-p)^R]\,. \tag{5}$$

<u>Proof</u>: The total expected number of sorties flown is A multiplied by the expected number of successful sorties flown by one aircraft. An aircraft flies zero successful sorties with

probability p--i.e., if the aircraft is killed on its first sortie. It flies R successful sorties with probability $(1-p)^R$--i.e., the aircraft survives all the sorties it flies. If R=1, the proposition follows forthwith. If $R \geq 2$, then for $1 \leq k < R$, an aircraft flies exactly k successful sorties with probability $(1-p)^k p$--i.e., the first k sorties are successful and the aircraft is killed on the $(k+1)^{st}$ sortie. The expected number of successful sorties an aircraft flies is thus

$$\sum_{k=1}^{R-1} k(1-p)^k p + R(1-p)^R .$$

The indicated sum can be evaluated by differentiating the formula for the sum of a geometric series and multiplying by the appropriate factors; after algebraic simplification, the proposition follows.

C. MAIN RESULTS

Throughout this section and Section D, for any nonnegative real x, $\lfloor x \rfloor$ will denote the integer part of x, <x> will denote the fractional part of x, and $\lceil x \rceil$ will denote the smallest integer greater than or equal to x.

1. Aircraft Kills

Theorems 1 through 6 consider a rectangular array of circles, R rows by A columns (R and A are positive integers). Figure 1 shows the example R=6, A=5. Let K be some positive integer that does not exceed RA. In each theorem, (the interiors of) exactly K circles in the array are colored black, in a random manner to be described. Theorems 1 through 6 each give formulas for the expected number of columns that contain at least one black circle, (i.e., a circle the interior of which has been colored black), under various assumptions about the distribution of the K black circles. (The term "white circle" indicates a circle the interior of which has not been colored black.)

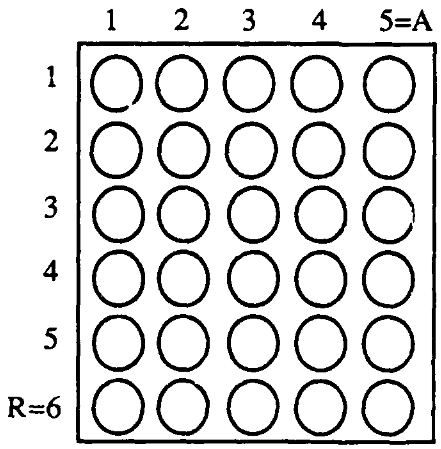

Figure 1. AN ARRAY OF CIRCLES

One can regard each column in the array described above as representing an aircraft with a sortie rate of R; there are A aircraft. Each circle in the array represents a sortie. Black circles represent sorties killed; the number (K) of such circles is extrinsically given. An aircraft is killed if it does not survive all of its sorties; this corresponds to at least one black circle appearing in the appropriate column of the array. Since Theorems 1 through 6 give the expected number of columns that contain at least one black circle, each of these theorems provides a distinct way to convert a number of sorties killed (K) to an expected number of aircraft killed. Theorems 1 and 3 below reproduce expressions (3) and (4) presented in Section A, but in the context of probabilistic experiments with the array of circles, which can be compared with other experiments.

Theorem 1. Assume that K is such that M = K/R is an integer. From the A columns choose M columns randomly and uniformly (so that any particular size-M subset of the A columns has an equal chance of being selected) and color each circle in each of these columns black. Then the expected number of columns with at least one black circle is

$$N_1(A,R,K) = K/R.$$

(Of course this result is obvious. But it puts the commonly used formula K/R for converting sorties killed to aircraft killed in the context of a probabilistic experiment, which can be compared to the experiments described in the theorems below.)

Theorem 2. (This generalizes Theorem 1 by considering any positive integer $K \leq RA$.) Let $I = \lfloor K/R \rfloor$. Choose I columns randomly and uniformly from the A columns and color each circle in each of these columns black. If I < A, then choose (randomly and uniformly) one additional column. From this column select K–RI circles randomly and uniformly and color each of these circles black. (If K/R is an integer, K–RI=0.) Then the expected number of columns with at least one black circle is

$$N_2(A,R,K) = \lceil K/R \rceil.$$

(Note that if K/R is an integer, then $N_2(A,R,K) = N_1(A,R,K)$.)

Theorem 3. Assume that K is such that $M = K/R$ is an integer. From each row, select M circles (in such a manner that each subset of M of the A circles in the row is equally likely to be chosen) and color them black. Treat different rows independently of one another. Then the expected number of columns with at least one black circle is

$$N_3(A,R,K) = A\left[1 - \left(1 - \frac{K}{RA}\right)^R\right].$$

(This is the same as equation (4) of Section A. Note that although it was derived for K/R an integer, the formula can be evaluated whether or not this is the case.)

Theorem 4. (This generalizes Theorem 3 by considering any positive integer $K \leq RA$.) Let $I = \lfloor K/R \rfloor$. From each row, choose I circles at random (as in Theorem 3) and color them black. Then select $K - IR$ rows at random (uniformly) from the R rows; from each of these rows choose (uniformly and independently of the other rows) one circle that has not already been colored black and color it black. Then the expected number of columns that contain at least one black circle is

$$N_4(A,R,K) = A[1-(1-\frac{\lfloor K/R \rfloor}{A})^{R-K+R\lfloor K/R \rfloor}(1-\frac{1+\lfloor K/R \rfloor}{A})^{K-R\lfloor K/R \rfloor}].$$

(Some special cases are as follows. If $K \geq R(A-1)$, then $\lfloor K/R \rfloor + 1 = A$, and $1 - (1 + \lfloor K/R \rfloor)/A$ is zero. If $K = R(A-1)$, interpreting 0^0 as 1 yields a consistent result. If $K > R(A-1)$, $N_4(A,R,K)$ becomes equal to A, as is consistent with the experiment. If K/R is an integer, then the experiment reduces to the experiment of Theorem 3, and $N_4(A,R,K)$ reduces to $N_3(A,R,K)$, as one would expect [with the proviso that 0^0 be interpreted as 1 in the case $K = R(A-1)$].)

Theorem 5. From the array of RA circles, choose K circles at random in such a manner that any size-K subset of the RA circles is equally likely to be chosen. It is clear that if $K > R(A-1)$, then each column will have at least one black circle. If $K \leq R(A-1)$, then the expected number of columns with at least one black circle is

$$N_5(A,R,K) = A\left(1 - \left[\binom{R(A-1)}{K}\Big/\binom{RA}{K}\right]\right).$$

Theorem 6. Choose $\lfloor K/A \rfloor$ rows at random (uniformly) from the R rows and color each circle in each of these rows black. If K/A is not an integer, then choose one additional row uniformly from those rows not previously chosen. From this row select $L = K - A\lfloor K/A \rfloor$ circles (L is less than A) at random (in such a manner that any subset of size L is equally likely to be chosen) and color them black. Then the expected number of columns with at least one black circle is given by

$$N_6(A,R,K) = \min(K,A) .$$

2. Expected Successful Sorties

Theorems 7 through 12 consider the experiments described in Theorems 1 through 6, respectively, and state formulas for the expected number of white circles in the array that are not located (vertically) under some black circle. In the analogy between circles and sorties, if black circles represent sorties killed, then white circles not located under a black circle represent "successful sorties," flown before the column (aircraft) encounters a black circle (is killed).

Theorem 7. For the experiment described in Theorem 1, the expected number of white circles not located (vertically) under some black circle is given by

$$S_1(A,R,K) = RA-K.$$

Theorem 8. For the experiment described in Theorem 2, the expected number of white circles not located (vertically) under some black circle is given by

$$S_2(A,R,K) = R(A-\lfloor K/R \rfloor-1) + \frac{R-K+R\lfloor K/R \rfloor}{K-R\lfloor K/R \rfloor+1} .$$

Theorem 9. For the experiment described in Theorem 3, the expected number of white circles not located (vertically) under some black circle is given by

$$S_3(A,R,K) = \frac{A}{K}(RA-K)\left[1-\left(1-\frac{K}{RA}\right)^R\right] .$$

Theorem 10. For the experiment described in Theorem 4, the expected number of white circles not located (vertically) under some black circle is

$$S_4(A,R,K) = A \sum_{n=1}^{R} \sum_{m=0}^{n} \frac{\binom{n}{m}\binom{R-n}{G-m}}{\binom{R}{G}} \left(1-\frac{I+1}{A}\right)^{m} \left(1-\frac{I}{A}\right)^{n-m} ,$$

where $I = \lfloor K/R \rfloor$, $G = K-RI$, and nonsensible combinatorial expressions[1] are regarded as zero.

Theorem 11. For the experiment described in Theorem 5, the expected number of white circles not located (vertically) under some black circle is

$$S_5(A,R,K) = \frac{A(RA-K)}{(K+1)} - A\frac{\binom{RA-R}{K+1}}{\binom{RA}{K}} .$$

If $K \geq R(A-1)$, the (numerator of the) second term should be regarded as zero.

Theorem 12. For the experiment described in Theorem 6, the expected number of white circles not located (vertically) under some black circle is

$$S_6(A,R,K) = \frac{A(R-\lfloor K/A \rfloor)}{1+\lfloor K/A \rfloor} - \frac{(R+1)(K-A\lfloor K/A \rfloor)}{(1+\lfloor K/A \rfloor)(2+\lfloor K/A \rfloor)} .$$

Note that if K/A is an integer, the second term is zero.

3. Inequalities Relating to Aircraft Kills

Recall the formulas representing expected numbers of aircraft killed that have been stated in Theorems 1 through 6 in Section 1, above:

$$N_1(A,R,K) = K/R ,$$

$$N_2(A,R,K) = \lceil K/R \rceil ,$$

$$N_3(A,R,K) = A[\, 1-(\, 1-\frac{K}{RA})^{R}\,] ,$$

---

1 I.e., expressions $\binom{N}{M}$ where $M < 0$, $N < 0$, and/or $M > N$.

$$N_4(A,R,K) = A[1 - (1 - \frac{\lfloor K/R \rfloor}{A})^{R-K+R\lfloor K/R \rfloor}(1 - \frac{1+\lfloor K/R \rfloor}{A})^{K-R\lfloor K/R \rfloor}],$$

$$N_5(A,R,K) = \begin{cases} A\left(1 - \left[\binom{R(A-1)}{K} \Big/ \binom{RA}{K}\right]\right) & \text{if } K \le R(A-1) \\ A & \text{if } K > R(A-1), \end{cases}$$

$$N_6(A,R,K) = \min(K,A).$$

This section states several inequalities regarding these formulas. Section C.4 provides a summary of these inequalities; proofs are given in Section D.

Although Theorems 1 through 6 were developed from probabilistic experiments which assumed that A, R, and K were positive integers, many of the formulas $N_i(A,R,K)$ can be evaluated for any positive real A, R, and K such that $K \le RA$ and $R \ge 1$. Similarly, although Theorems 1 and 3 were developed assuming that K/R was a positive integer, formulas $N_1(A,R,K)$ and $N_3(A,R,K)$ can be evaluated whether or not this is the case. For conciseness in stating the theorems, two "assumption sets" will be used, defined thus:

<u>Assumption Set 1</u>: A, R, and K are positive real numbers such that $K \le RA$ and $R \ge 1$;

<u>Assumption Set 2</u>: A, R, and K are positive integers such that $K \le RA$.

Several theorems need the stronger Assumption Set 2, as noted in the statement of the theorems.

First, note the trivial results that for any A, R, and K that satisfy Assumption Set 1,

$$N_1(A,R,K) \le N_2(A,R,K)$$

and

$$N_1(A,R,K) \le N_6(A,R,K)$$

(i.e., $K/R \le \lceil K/R \rceil$ and $K/R \le \min(K,A)$). Further results are stated as Theorems 13 through 19, below.

<u>Theorem 13</u>. For any A, R, and K that satisfy Assumption Set 2, $N_2(A,R,K) \le N_6(A,R,K)$. (This is not necessarily true under Assumption Set 1.)

Theorem 14. For any A, R, and K that satisfy Assumption Set 1, $N_3(A,R,K) \leq N_6(A,R,K)$.

Theorem 15. For any A, R, and K that satisfy Assumption Set 2,

$$N_4(A,R,K) \leq N_6(A,R,K)$$

and

$$N_5(A,R,K) \leq N_6(A,R,K).$$

($N_5(A,R,K)$, and sometimes $N_4(A,R,K)$, are not defined for noninteger A, R, and K.)

Theorem 16. For any A, R, and K that satisfy Assumption Set 1, $N_1(A,R,K) \leq N_3(A,R,K)$.

Theorem 17. For any A, R, and K that satisfy Assumption Set 2, $N_2(A,R,K) \leq N_4(A,R,K)$. (If K/R is an integer, then of course $N_2(A,R,K)$ reduces to $N_1(A,R,K)$ and $N_4(A,R,K)$ reduces to $N_3(A,R,K)$. It is not necessarily true that $N_2(A,R,K) \leq N_3(A,R,K)$ if K/R is not an integer.)

Theorem 18. For any A, R, and K that satisfy Assumption Set 2, $N_3(A,R,K) \leq N_4(A,R,K)$. (The theorem is also true under Assumption Set 1 plus the additional condition that $\lfloor K/R \rfloor + 1 \leq A$. That is, under these conditions, $N_4(A,R,K)$ is defined and $N_3(A,R,K) \leq N_4(A,R,K)$.)

Theorem 19. For any A, R, and K that satisfy Assumption Set 2, $N_4(A,R,K) \leq N_5(A,R,K)$. ($N_5(A,R,K)$ for its very definition assumes that A, R, and K are integers, and the proofs given in Section D utilize the integrality of A, R, and K.) The proofs treat the following cases separately:

(i) $K > R(A-1)$ (in this trivial case, $N_4(A,R,K) = N_5(A,R,K) = A$),

(ii) K/R is an integer and $K \leq R(A-1)$,

(iii) $K < R$ and $K \leq R(A-1)$, and

(iv) K/R is not an integer and $R < K < R(A-1)$.

4. Summary of Inequalities Relating to Aircraft Kills

The results of Section C.3 can be summarized as follows (a review of the formulas $N_i(A,R,K)$ and the assumption sets stated at the beginning of Section C.3 will be helpful).

(a) For any A, R, and K satisfying Assumption Set 1,

$$N_1(A,R,K) \leq N_2(A,R,K)$$

and

$$N_1(A,R,K) \leq N_3(A,R,K) \leq N_6(A,R,K).$$

Under Assumption Set 1: 1) there is no clear inequality relationship between $N_2(A,R,K)$ and $N_3(A,R,K)$, i.e., depending on the values of A, R, and K, $N_2(A,R,K)$ might be less than, or equal to, or greater than $N_3(A,R,K)$; 2) the same is true of $N_2(A,R,K)$ and $N_6(A,R,K)$; and 3) $N_4(A,R,K)$ and $N_5(A,R,K)$ are (in general) undefined.

(b) For any A, R, and K satisfying Assumption Set 2,

$$N_1(A,R,K) \leq N_2(A,R,K) \leq N_4(A,R,K) \,,$$

$$N_1(A,R,K) \leq N_3(A,R,K) \leq N_4(A,R,K) \,,$$

and

$$N_4(A,R,K) \leq N_5(A,R,K) \leq N_6(A,R,K) \,.$$

Under Assumption Set 2, there is no clear inequality relationship between $N_2(A,R,K)$ and $N_3(A,R,K)$, unless K/R is an integer, as indicated below.

(c) For any A, R, and K such that Assumption Set 2 is satisfied and also K/R is an integer,

$$N_1(A,R,K) = N_2(A,R,K) \,,$$

$$N_2(A,R,K) \leq N_3(A,R,K) \,,$$

$$N_3(A,R,K) = N_4(A,R,K) \,,$$

and

$$N_4(A,R,K) \leq N_5(A,R,K) \leq N_6(A,R,K) \,.$$

5. <u>Inequalities Relating to Expected Successful Sorties</u>

Recall the three formulas developed in Theorems 7, 8, and 9 (see the discussion in Section C.2, above, for interpretations of these formulas):

$$S_1(A,R,K) = RA-K$$

$$S_2(A,R,K) = R(A-\lfloor K/R \rfloor-1) + \frac{R-K+R\lfloor K/R \rfloor}{K-R\lfloor K/R \rfloor+1} \ ,$$

$$S_3(A,R,K) = \frac{A}{K}(RA-K)\left[1-(1-\frac{K}{RA})^R \right] \ .$$

Theorem 20. For any positive real A, R, and K such that $K \leq RA$ and $R \geq 1$ (i.e., Assumption Set 1),

$$S_2(A,R,K) \leq S_1(A,R,K) \ .$$

Theorem 21. For any positive real A, R, and K such that $K \leq RA$ and $R \geq 1$,

$$S_3(A,R,K) \leq S_1(A,R,K).$$

Depending on the values of A, R, and K, $S_2(A,R,K)$ may be greater than, equal to, or less than $S_3(A,R,K)$. In numerical evaluations of the functions for several thousand (integer) value combinations of A, R, and K, the relationships

$$S_2(A,R,K) \geq S_4(A,R,K) \ ,$$

$$S_3(A,R,K) \geq S_4(A,R,K) \ ,$$

and

$$S_4(A,R,K) \geq S_5(A,R,K) \geq S_6(A,R,K)$$

always occurred. However, this paper does not present proofs of these hypotheses; it is possible that some or all of them are false.

D. PROOFS OF RESULTS

1. Aircraft Kills

Theorems 1 through 6 are proved below.

(1) Exactly M columns have black circles in them (and these columns are completely filled with black circles); the rest of the columns have no black circles. Thus the expected number of columns with at least one black circle is M, which equals K/R.

(2) If K/R is an integer, the reasoning in Theorem 1 applies. If not, then (recalling that $I = \lfloor K/R \rfloor$) I columns are completely filled with black circles and one additional column has K–RI black circles; if K/R is not an integer, $K-RI \geq 1$. Thus the total (and expected) number of columns with at least one black circle, if K/R is not an integer, is I+1.

Thus overall, the expected (total) number of columns with at least one black circle is $\lceil K/R \rceil$.

(3) The expected number of columns with at least one black circle is the number of columns (A) multiplied by the probability that a particular column contains at least one black circle--which is one minus the probability it contains no black circles. This latter probability is the product (because different rows are treated independently of one another) of the probabilities that a particular row does not have a black circle in the particular column being considered. Each of these probabilities is

$$1 - \binom{A-1}{M-1} \Big/ \binom{A}{M} = 1 - M/A = 1 - K/(RA)\,.$$

The desired formula follows forthwith.

(4) Recall that $I = \lfloor K/R \rfloor$. If K/R is an integer, this theorem reduces to Theorem 3, so assume that K/R is not integer. Since A, K, and R are integers and K/R is assumed to be less than or equal to A, then $I \le A-1$. (Thus all combinatorial expressions appearing below are sensible.) A row with I circles will have a black circle at a particular column location with probability

$$\binom{A-1}{I-1} \Big/ \binom{A}{I} = \frac{I}{A}\,.$$

A row with I+1 circles will have a black circle at a particular column location with probability

$$\binom{A-1}{I} \Big/ \binom{A}{I+1} = \frac{I+1}{A}\,.$$

There are R–K+IR rows with I black circles each and K–IR rows with I+1 black circles each. (It can be verified that these formulas imply that the total number of black circles is K, as it should be.) Since different rows have circles chosen independently, the probability a particular column contains no black circles is

$$\left(1-\frac{I}{A}\right)^{R-K+IR} \left(1-\frac{I+1}{A}\right)^{K-IR}.$$

(If I + 1 = A, i.e., K > R(A–1), the above expression is zero, as it should be.) As in

Theorem 3, A multiplied by one minus this probability is the expected number of columns with at least one black circle. Substituting $\lfloor K/R \rfloor$ for I yields the desired formula.

(5) There are $\binom{RA}{K}$ (equally likely) ways that the K black circles can be chosen. Consider any particular column--it has R circles. The number of ways the K black circles can be chosen so that none of these R circles is black is $\binom{RA-R}{K}$. Thus, the probability that a particular column contains no black circle is

$$\binom{R(A-1)}{K} \Big/ \binom{RA}{K}.$$

As in Theorem 3, the expected number of columns with at least one black circle is A multiplied by one minus the above probability; the desired formula follows forthwith.

(6) If $\lfloor K/A \rfloor \geq 1$ (i.e., if $K \geq A$) then at least one row will be completely filled with black circles and all A columns will contain at least one black circle. If $K < A$, then $\lfloor K/A \rfloor = 0$ and $K - A\lfloor K/A \rfloor = K$; one row is filled with K circles, exactly K columns contain exactly one black circle each, and the remaining columns contain no black circles. The desired result follows forthwith.

2. Expected Successful Sorties

Theorems 7 through 12 are proved below.

(7) Let $M = K/R$; M is assumed to be integer. The white circles not under a black circle are precisely those white circles in columns that contain no black circle. There are $A-M$ such columns; each has R white circles. The formula follows forthwith.

(8) Let $I = \lfloor K/R \rfloor$. If K/R is an integer, then the process is like the process of Theorem 1 and thus the formula of Theorem 7 will apply. Theorem 8 will thus first be proved assuming that K/R is not an integer, and it will then be shown that if K/R is an integer, the formula of Theorem 8 reduces to the formula of Theorem 7. Of course the assumption is still made that K, R, and A are positive integers and $K \leq RA$. If K/R is not an integer, these assumptions imply that $I \leq A-1$ and $1 \leq K-RI \leq R-1$.

The process described in Theorem 2 (which is used for Theorem 8) results in I columns completely filled with black circles, $A-I-1$ columns that are completely white circles, and one column that contains $G=K-RI$ black circles. Let the random variable N be the number of white circles in this column that are not located under a black circle. N can assume values between zero and R–G, inclusive. The expected value of N is

$$E[N] = \sum_{n=1}^{\infty} P(N \geq n) = \sum_{n=1}^{R-G} P(N \geq n) .$$

For each n from 1 through R–G, $N \geq n$ if and only if the top n circles of the column are all white. The probability of this equals the number of choices of G of the R circles where the top n circles are white divided by the total number of ways of choosing G of the R circles in the column. I.e.,

$$P(N \geq n) = \binom{R-n}{G} \Big/ \binom{R}{G} .$$

Then

$$E[N] = \left[1 \Big/ \binom{R}{G} \right] \sum_{n=1}^{R-G} \binom{R-n}{G} .$$

It can be shown by standard combinatorial methods that the indicated sum in the above expression is equal to $\binom{R}{G+1}$; this yields

$$E[N] = \frac{R-G}{G+1} = \frac{R-K+RI}{K-RI+1} .$$

(Note that E[N] might or might not be greater than one, depending on the relative values of R and G.) The overall expected number of white circles in the array that are not located under some black circle is then E[N]+R(A–I–1), i.e.,

$$\frac{R-K+RI}{K-RI+1} + R(A-I-1) ,$$

which equals the formula $S_2(A,R,K)$ as stated. If K/R is an integer then K=RI and the formula above becomes equal to the formula RA–K developed in Theorem 7.

(9) Consider any particular column. The circle at any particular row location will be black with probability $p = M/A = K/(RA)$. Different rows are treated independently. Thus the probability that the column contains exactly k white circles at first and then a black circle is

$$(1-p)^k p \qquad\qquad k=1,\ldots,R-1,$$

and the probability that the column contains only white circles (i.e., R white circles) is

$$(1-p)^R.$$

Thus, the expected number of white circles in a column that are not located under a black circle is

$$\sum_{k=1}^{R-1} k(1-p)^k p + R(1-p)^R .$$

By the Proposition in Section B, this formula equals

$$\frac{1-p}{p}\,[1-(1-p)^R].$$

The expected number of white circles in the whole array that are not located under a black circle is then A times the number in any particular column (even though different columns are not independent). Replacing p by K/(RA), the result is $S_3(A,R,K)$ as stated, namely

$$\frac{A}{K}(RA-K)\,[\,1-(1-\frac{K}{RA})^R\,] .$$

The proofs of Theorems 10, 11, and 12 all use the following notation. Consider a specific column, call it column j, and let the (integer-valued) random variable N represent the number of white circles located at the top of column j. That is, for $0 \le n \le R-1$, N=n precisely when column j has a white circle in rows 1 through n and a black circle in row n+1; N=R means that column j contains only white circles. At the outset, by symmetry, all columns are treated identically, thus the overall expected number of white circles not located (vertically) under some black circle is equal to AE[N]. Also,

$$E[N] = \sum_{n=1}^{R} P(N \ge n) .$$

Each proof derives the appropriate formulas for $P(N \ge n)$, evaluates the sum E[N], and multiplies by A to obtain the final formula $S_i(A,R,K)$ (for i = 4, 5, and 6, in turn).

(10) Let $I = \lfloor K/R \rfloor$ and $G = K-RI$. Any realization of the experiment results in G "dark rows," each containing (exactly) I+1 black circles, and R–G "light rows," each containing I black circles. The locations of the dark rows are a size-G random sample from {1,...,R}. To find $P(N \ge n)$ (for n=1,...,R) define the following events:

W_n--For each (all) of the top n rows of the array of circles, the circle in column location j is white,

D_{nm}--Exactly m of the top n rows of the array are "dark" (and the remaining n–m rows are "light") (defined for m=0,...,n).

Note that W_n is simply the event $N \geq n$, as described above. A "light" row has a white circle in column location j with probability $1 - I/A$ (i.e., $\binom{A-1}{I} \Big/ \binom{A}{I}$) and a "dark" row has a white circle in column location j with probability $1 - (I+1)/A$. (By the assumptions, $I \leq A-1$ unless K=RA, in which case the entire array consists of black circles. Let us not consider this vacuous case until the end of the proof.) Furthermore, the events that different rows have white circles in column j are independent. Thus

$$P(W_n \mid D_{nm}) = \left(1 - \frac{I+1}{A}\right)^m \left(1 - \frac{I}{A}\right)^{n-m}.$$

The number of dark rows in the top n rows of the array follows a hypergeometric distribution, i.e.,

$$P(D_{nm}) = \frac{\binom{n}{m}\binom{R-n}{G-m}}{\binom{R}{G}}.$$

(Of course D_{nm} will not occur if $m > n$, or $m > G$, or $m < n-(R-G)$. But in these cases, one of the combinatorial expressions in the numerator of the above expression will not make sense, and should be interpreted as zero.) Thus $P(W_n) = P(N \geq n)$ equals

$$\sum_{m=0}^{n} P(W_n \mid D_{nm})\, P(D_{nm}).$$

This sum can be put in the form of the z-transform of the hypergeometric distribution, and does not appear to have a closed form expression. Further summing on n and multiplying by A yields the formula $S_4(A,R,K)$ stated in the theorem.

If K/R is an integer, then G is zero and only the m=0 term in the inner sum is counted. Substituting K/R for I, $S_4(A,R,K)$ then reduces to

$$A \sum_{n=1}^{R} \left(1 - \frac{K}{RA}\right)^n ;$$

evaluating this geometric series yields the formula $S_3(A,R,K)$. In the further special case K=RA (continuing to assume that K, R, and A are positive integers), this formula is zero (as the expected number of successful sorties should be).

(11) This proof treats the cases $K \geq R(A-1)$ and $K < R(A-1)$ separately. With suitable interpretation, the formula developed in the latter case will then be found to be correct for the former case also.

If $K \geq R(A-1)$, then N can assume values between zero and RA–K, inclusive. By the nature of the experiment

$$P(N \geq n) = \binom{RA-n}{K} \Big/ \binom{RA}{K} \qquad n=0,\ldots,RA-K.$$

If K = RA, E[N] is clearly zero. Otherwise (i.e., if $R(A-1) \leq K < RA$), then

$$E[N] = \left[1 \Big/ \binom{RA}{K} \right] \sum_{n=1}^{RA-K} \binom{RA-n}{K}.$$

By standard combinatorial methods (as in the proof of Theorem 8) the indicated sum is equal to $\binom{RA}{K+1}$. Substituting, simplifying, and multiplying by A yields that the overall expected number of white circles not located under some black circle equals

$$A(RA-K)/(K+1).$$

This formula also makes sense for the case K=RA.

If $K < R(A-1)$, then N can assume values between zero and R, inclusive. For $n=0,\ldots,R$, $P(N \geq n)$ is the same as given above, thus

$$E[N] = \left[1 \Big/ \binom{RA}{K} \right] \sum_{n=1}^{R} \binom{RA-n}{K}.$$

To evaluate the indicated sum, first note that for any positive integers G and M,

$$\sum_{m=0}^{M-1} \binom{G+m}{G} = \binom{G+M}{G+1}.$$

(This result can be proved by induction on M.) Substituting m=RA–K–n in the expression for E[N] results in the indicated sum being equal to

$$\sum_{m=RA-K-R}^{RA-K-1} \binom{K+m}{K},$$

which by the above result equals

$$\binom{RA}{K+1} - \binom{RA-R}{K+1}.$$

After simplification, AE[N] (i.e., the expected number of white circles not located under any black circle) is then

$$\frac{A(RA-K)}{(K+1)} - A\frac{\binom{RA-R}{K+1}}{\binom{RA}{K}},$$

which is formula $S_5(A,R,K)$. Note that the first term of this expression is the same as the expected number of white circles not located under any black circle in the case $K \geq R(A-1)$. Also, if $K \geq R(A-1)$, the numerator of the second term does not make sense and should be regarded as zero. Thus the formula $S_5(A,R,K)$ is correct for all values of K (between 1 and RA, inclusive).

(12) Let $C = \lfloor K/A \rfloor$. The two cases K/A an integer and K/A not an integer will be treated separately. The formula developed in the latter case will then be shown to be correct in the former case also.

If K/A is an integer, then C = K/A. Assume that $C \leq R-1$; the vacuous case K = RA will be treated presently. Any realization of the experiment results in C rows of the array completely filled with black circles, with the remaining R–C rows having no black circles. The locations of the C "complete" rows are a size–C random sample from {1,...,R}. The event N≥n will occur precisely when all C complete rows are located (strictly) below row n. Thus

$$P(N\geq n) = \binom{R-n}{C} \Big/ \binom{R}{C}$$

for n=0,...,R–C; it is clear that N cannot assume values strictly greater than R–C. Then

$$E[N] = \left[1 \Big/ \binom{R}{C}\right] \sum_{n=1}^{R-C} \binom{R-n}{C}.$$

As in the proofs of Theorems 8 and 11, the indicated sum equals $\binom{R}{C+1}$. After simplification, AE[N] becomes equal to

$$A(R-C)/(1+C).$$

If K/A is not an integer, then let L = K–AC (where $C = \lfloor K/A \rfloor$). It is clear that L is (an integer) between 1 and A–1, inclusive. The experiment results in C "complete" rows as before, one "incomplete" row containing L black circles, and R–C–1 rows containing no black circles. (It is clear that $C \le R-1$.) The random variable N can (again) take on values between zero and R–C, inclusive. For each n in this range, define the following events:

T_n--all complete rows are located strictly below row n,

I_n--the incomplete row is located strictly below row n,

$\bar{I}_n$--the complement of I_n, i.e., the incomplete row is located at row n or above, and

V_j--the incomplete row has a white circle at column location j.

The event $N \ge n$ is the disjoint union of the events T_nI_n and $T_n\bar{I}_nV_j$. By the nature of the experiment V_j is independent of I_n and T_n (for all n), and

$$P(V_j) = 1 - \left[\binom{A-1}{L} \Big/ \binom{A}{L}\right]$$
$$= 1 - L/A.$$

Since the complete and incomplete rows are randomly (uniformly) chosen from the R rows,

$$P(T_n) = \binom{R-n}{C} \Big/ \binom{R}{C} \qquad n=1,\ldots,R-C.$$

(T_n cannot occur for $n > R-C$.) Given T_n, then of the R–C rows that are not complete, R–n–C are located strictly below row n. Since the "incomplete" row is selected uniformly from the R–C "not complete" rows,

$$P(I_n | T_n) = \frac{R-n-C}{R-C} .$$

Combining the above results yields

$$P(N \geq n) = P(T_n I_n) + P(T_n \bar{I}_n V_j)$$

$$= \frac{R-n-C}{R-C} \frac{\binom{R-n}{C}}{\binom{R}{C}} + \frac{n}{R-C} \frac{\binom{R-n}{C}}{\binom{R}{C}} \left(1 - \frac{L}{A}\right) .$$

Then

$$E[N] = \sum_{n=1}^{R-C} P(N \geq n) = \left[1 \Big/ \binom{R}{C}\right] \left[\sum_{n=1}^{R-C} \binom{R-n}{C} - \frac{L}{A(R-C)} \sum_{n=1}^{R-C} n \binom{R-n}{C}\right] .$$

As indicated in the proofs of Theorems 8 and 11,

$$\sum_{n=1}^{R-C} \binom{R-n}{C} = \binom{R}{C+1} .$$

The second indicated sum can be evaluated by setting, for each n,

$$n \binom{R-n}{C} = \sum_{m=1}^{n} \binom{R-n}{C} ,$$

interchanging the order of summation, and utilizing the technique outlined in the proof of Theorem 11. The sum turns out to be equal to $\binom{R+1}{C+2}$. After simplification,

$$E[N] = \frac{R-C}{C+1} - \frac{L(R+1)}{A(C+2)(C+1)} .$$

Multiplying by A and substituting $\lfloor K/A \rfloor$ for C and $K-A\lfloor K/A \rfloor$ for L yields the formula $S_6(A,R,K)$ as stated.

Note that if K/A is an integer, then L as defined above is zero. In this case, the second term of $S_6(A,R,K)$ is zero and first term, $A(R-C)/(C+1)$, is the formula for AE[N] that was derived earlier for the case K/A an integer. If K=RA, the whole expression is zero (as the expected number of successful sorties should be). Thus the formula $S_6(A,R,K)$ is correct for all values of K between zero and RA.

3. Inequalities Relating to Aircraft Kills

Theorems 13 through 19 are proved below.

(13) It is assumed that $K \le RA$, so $K/R \le A$. Since A is assumed to be integer, $\lceil K/R \rceil \le A$. Since R is assumed to be a positive integer, then $R \ge 1$, and thus $K/R \le K$. Since K is assumed to be integer, then $\lceil K/R \rceil \le K$. Thus $\lceil K/R \rceil$, and, of course, K/R, are less than or equal to min(K,A).

(14) It is desired to prove that $N_3(A,R,K) \le N_6(A,R,K)$, i.e.,

$$A\left[1-\left(1-\frac{K}{RA}\right)^R\right] \le \min(K,A)\,. \tag{6}$$

If $A \le K$, the theorem is clearly true, as Assumption Set 1 implies that the term in brackets on the LHS of (6) is less than or equal to 1. For the rest of this proof assume that $K < A$; the theorem is then equivalent to the statement

$$\left(1-\frac{K}{RA}\right)^R \ge 1-\frac{K}{A}\,. \tag{7}$$

Consider the function

$$f(x) = \left[1-\frac{(K/A)}{x}\right]^x \,.$$

It is clear that f(x) is sensible for $x \ge 1$ and that

$$f(1) = 1-\frac{K}{A}\,.$$

If it can be shown that f(x) is a nondecreasing function of x for $x \ge 1$, then since $R \ge 1$, inequality (7) and the theorem will follow. To show that f(x) is nondecreasing, it will suffice to show that the derivative f'(x) is nonnegative for $x \ge 1$. By standard procedures, it can be shown that

$$f'(x) = \left(1-\frac{K}{Ax}\right)^{x}\left[-\ln\left(\frac{Ax}{Ax-K}\right) + \frac{K}{Ax-K}\right].$$

Since it is assumed that $K < A$ and $x \geq 1$, the first term in the above expression is strictly positive and thus $f'(x)$ is nonnegative if and only if the term in brackets is nonnegative. Let

$$y = \frac{Ax}{Ax-K} ;$$

y is greater than one, and the nonnegativity of $f'(x)$ follows from the elementary inequality

$$\ln y \leq y-1 \qquad \text{for } y \geq 1 .$$

(15) It is clear from inspection that all the formulas under consideration do not exceed A. If $K < A$--i.e., the number of black circles in the array is less than the number of columns--then in any realization of the experiments described in Theorems 4 and 5, at most K columns will have at least one black circle. The expected number of columns with at least one black circle is thus less than or equal to K.

(16) Since K, R, and A are positive and $K \leq RA$, $K/(RA)$ and $1 - K/(RA)$ are quantities that lie in the interval [0,1]. Since $R \geq 1$,

$$1-\frac{K}{RA} \geq \left(1-\frac{K}{RA}\right)^{R}$$

which implies

$$\frac{K}{RA} \leq 1-\left(1-\frac{K}{RA}\right)^{R}$$

which implies

$$\frac{K}{R} \leq A\left[1-\left(1-\frac{K}{RA}\right)^{R}\right] \qquad \text{QED.}$$

(17) Let $I = \lfloor K/R \rfloor$. If K/R is an integer, this theorem reduces to Theorem 16 so assume that K/R is not an integer. Then $\lceil K/R \rceil = I+1$ and the quantity $K-RI$ is greater than zero; since it is an integer, it is greater than or equal to 1. The statement that $N_2(A,R,K) \leq N_4(A,R,K)$ is then equivalent to

$$\left(1-\frac{I}{A}\right)^{R-K+RI}\left(1-\frac{I+1}{A}\right)^{K-RI} \leq \left(1-\frac{I+1}{A}\right). \qquad (8)$$

By Theorem 13, $I+1 \leq A$, so $1 - (I+1)/A$ is an element of [0,1], and since $K-RI \geq 1$,

$$\left(1-\frac{I+1}{A}\right)^{K-RI} \leq \left(1-\frac{I+1}{A}\right).$$

Since $K \leq RA$, the first term on the LHS of (8) is also an element of $[0,1]$, and (8) follows forthwith.

(18) If K/R is an integer, then $\lfloor K/R \rfloor = K/R$ and $N_3(A,R,K) = N_4(A,R,K)$. Thus assume that K/R is not an integer and let I and g denote the integer and fractional parts of K/R, respectively (so $K/R = I + g$); note that $I+1 \leq A$. Then $N_4(A,R,K)$ equals

$$A\left[1-\left(1-\frac{I}{A}\right)^{R(1-g)}\left(1-\frac{I+1}{A}\right)^{Rg}\right].$$

Recall that $N_3(A,R,K)$ is

$$A\left[1-\left(1-\frac{K}{RA}\right)^{R}\right].$$

To prove that $N_3(A,R,K) \leq N_4(A,R,K)$ it will suffice to prove that

$$\left(1-\frac{K}{RA}\right) \geq \left(1-\frac{I}{A}\right)^{1-g}\left(1-\frac{I+1}{A}\right)^{g}.$$

This is clearly true if $I+1 = A$; otherwise it is equivalent to the statement

$$\ln\left(1-\frac{K}{RA}\right) \geq (1-g)\ \ln\left(1-\frac{I}{A}\right) + g\ \ln\left(1-\frac{I+1}{A}\right).$$

This latter condition follows from the concavity of the function $\ln x$ and the fact that

$$1-\frac{K}{RA} = (1-g)\left(1-\frac{I}{A}\right) + g\left(1-\frac{I+1}{A}\right).$$

(19) The four cases are treated separately, in turn.

(i) In the trivial case $K > R(A-1)$, $N_4(A,R,K) = N_5(A,R,K) = A$, as stated in the theorem.

From now on, assume that $K \leq R(A-1)$ so that $N_5(A,R,K)$ is defined by the first formula given. Then $N_4(A,R,K) \leq N_5(A,R,K)$ is equivalent to the statement

$$\binom{R(A-1)}{K} \Big/ \binom{RA}{K} \le \left(1-\frac{I}{A}\right)^{R-K+RI} \left(1-\frac{I+1}{A}\right)^{K-RI}, \tag{9}$$

where, as before, $I = \lfloor K/R \rfloor$. The left hand side of (9) is expressible as

$$\frac{(RA-R)!\,(RA-K)!}{(RA)!\,(RA-R-K)!} \tag{10}$$

which can be expressed as

$$\prod_{m=0}^{R-1} \left(\frac{RA-K-m}{RA-m}\right) \tag{11}$$

and can also be expressed as

$$\prod_{n=0}^{K-1} \left(\frac{RA-R-n}{RA-n}\right). \tag{12}$$

In the right hand side of expression (9), $1 - I/A = (RA-RI)/(RA)$, and $1 - (I+1)/A = (RA-RI-R)/(RA)$.

(ii) If K/R is an integer, then $K = RI$. Using expression (11), expression (9) is then equivalent to

$$\prod_{m=0}^{R-1} \left(\frac{RA-K-m}{RA-m}\right) \le \left(\frac{RA-K}{RA}\right)^{R}. \tag{13}$$

By the assumptions on A, R, and K, each term of the indicated product is a proper fraction, and the ratio of two positive integers, thus for each m from 0 to $R-1$,

$$0 < \frac{RA-K-m}{RA-m} \le \frac{RA-K}{RA}.$$

Taking the product over m from zero through $R-1$ yields the desired result.

(iii) If $K < R$ then $I = 0$. Using expression (12), expression (9) is then equivalent to

$$\prod_{n=0}^{K-1} \left(\frac{RA-R-n}{RA-n}\right) \le \left(\frac{RA-R}{RA}\right)^{K}. \tag{14}$$

Since it is assumed that $K \le R(A-1)$, then for each value of n from 0 through $K-1$, $(RA-R-n)/(RA-n)$ is a proper fraction; this fraction is less than or equal to $(RA-R)/(RA)$. Statement (14) follows forthwith.

(iv) *Using expression (11), expression (9) is equivalent to the statement*

$$\prod_{m=0}^{R-1}\left(\frac{RA-K-m}{RA-m}\right) \le \left(\frac{RA-RI}{RA}\right)^{R-K+RI}\left(\frac{RA-RI-R}{RA}\right)^{K-RI},$$

which is equivalent to

$$\left[\prod_{m=0}^{R-1}(RA-K-m)\right]\Big/\left[(RA-RI)^{R-K+RI}(RA-RI-R)^{K-RI}\right] \le \prod_{m=0}^{R-1}\left(\frac{RA-m}{RA}\right). \quad (15)$$

Recall the elementary fact that for positive (real) quantities b and d, and nonnegative real a and c,

$$\left.\begin{array}{r} a \le b \\ \text{and } a \le c \\ \text{and } b-a = d-c \end{array}\right\} \Rightarrow \frac{a}{b} \le \frac{c}{d} \qquad (16)$$

The proof proceeds by constructing, for each i between zero and $R-1$, inclusive, quantities a_i, b_i, c_i and d_i that satisfy the constraints on the LHS of (16). It will also be shown that $b_i - a_i = i$, for each i, and that

Πa_i is the numerator of the LHS of (15),

Πb_i is the denominator of the LHS of (15), and

$\Pi(c_i/d_i)$ is the RHS of (15),

where all indicated products are from $i = 0$ through $R-1$. The desired inequality (15) will then follow forthwith from the "elementary fact." To start, let $c_i = RA-i$ and $d_i = RA$, for $i = 0$ through $R-1$; note that $d_i - c_i = i$, for all such i.

Consider the exponents of the terms in the denominator of the LHS of (15). It is assumed that K/R is not an integer, thus $K/R > I$ and $K-RI \ge 1$. But also $K-RI = R\langle K/R\rangle$, and thus is strictly less than R, as $\langle K/R\rangle$ is strictly between zero and one. Thus $K-RI \le R-1$ and $R-K+RI \ge 1$. Thus both of the indicated exponents are positive integers. Since $K < R(A-1)$ and $RI < K$, the quantities $(RA-RI)$ and $(RA-RI-R)$ also are positive integers.

Let (the nonnegative integer) $\overline{m}$ be such that the term RA–K–$\overline{m}$ in the numerator of the LHS of (15) is equal to RA–RI–R. It is clear that such an $\overline{m}$ exists, namely

$$\overline{m} = R - (K - RI).$$

It is clear from the properties of K–RI that $\overline{m} > 0$ and $\overline{m} \leq R-1$. The cases $\overline{m} = R-1$ and $\overline{m} < R-1$ are treated separately.

Case (a): $\overline{m} = R-1$

This implies that K–RI = 1, and thus the LHS of (15) becomes

$$\left(\frac{RA-K-(R-1)}{RA-RI-R}\right) \prod_{m=0}^{R-2} \left(\frac{RA-K-m}{RA-RI}\right). \tag{17}$$

Further, the numerator and denominator of the first term of (17) are equal; in the mth term of the indicated product (for m=0,...,R–2), the denominator-minus-numerator difference is

$$(RA-RI) - (RA-K-m)$$

which equals

$$(K-RI) + m$$

or

$$1 + m \ .$$

Thus, let

$$\begin{aligned}
a_0 &= RA - K - (R-1) && \\
b_0 &= RA - RI - R && \\
a_i &= RA - K - (i-1) && i=1,\ldots,R-1 \\
b_i &= RA - RI && i=1,\ldots,R-1
\end{aligned}$$

and apply the construction (16) for each i.

Case (b): $\overline{m} \leq R-2$

In this case, the LHS of (15) can be written as

$$\underbrace{\left[\prod_{m=0}^{\overline{m}-1} \left(\frac{RA-K-m}{RA-RI}\right)\right]}_{\overline{m}\text{ terms}} \underbrace{\left(\frac{RA-K-\overline{m}}{RA-RI-R}\right)}_{1\text{ term}} \underbrace{\prod_{m=\overline{m}+1}^{R-1} \left(\frac{RA-K-m}{RA-RI-R}\right)}_{R-(\overline{m}+1)\text{ terms}}. \tag{18}$$

By the definition of $\overline{m}$, the numerator and denominator of the middle term of (18) are equal (and are nonzero). Since RI < K, RA–K–m < RA–RI for any nonnegative integer m, thus each term in the first indicated product of (18) is a proper fraction (and each numerator is positive, since K ≤ RA–R). For the m^{th} term in this product, the denominator-minus-numerator difference

$$(RA-RI) - (RA-K-m)$$

equals

$$(K-RI) + m\,;$$

as m varies from 0 through $\overline{m}-1$, this difference assumes values of K–RI through R–1, successively.

Now consider the second indicated product of (18). If $m > \overline{m}$, then by the definition of $\overline{m}$

$$m > R-K+RI$$

thus

$$RA-K-m < RA-RI-R$$

so each term in the indicated product is a proper fraction (and, again, each numerator is a positive integer). The denominator-minus-numerator difference can be shown to equal

$$m - \overline{m}\ ;$$

as m varies from $\overline{m}+1$ through R–1, this difference takes on the values 1 through (K–RI) – 1, successively. (Since it is assumed that $\overline{m} \le R-2$, K–RI ≥ 2.) Therefore, let

$$a_0 = RA - K - \overline{m} = RA - RI - R$$

$$b_0 = RA - RI - R$$

$$\left.\begin{aligned} a_i &= RA - K - (\overline{m}+i) \\ b_i &= RA - RI - R \end{aligned}\right\} \quad i=1,\ldots,R-(\overline{m}+1)$$

$$\left.\begin{aligned} a_i &= RA - K - (i-[K-RI]) \\ b_i &= RA - RI \end{aligned}\right\} \quad i=R-\overline{m},\ldots,R-1$$

(note that $R-\overline{m} = K-RI$) and apply the construction (16) for each i.

4. Inequalities Relating to Expected Successful Sorties

Theorems 20 and 21 are proved below.

(20) Let $I = \lfloor K/R \rfloor$. Recall that

$$S_2(A,R,K) = R(A-I-1) + \frac{R-K+RI}{K-RI+1} \ .$$

By the definition of I, the indicated fraction in the expression for $S_2(A,R,K)$ is less than or equal to R–K+RI, and thus

$$S_2(A,R,K) \leq R(A-I-1) + R-K+RI = RA-K = S_1(A,R,K) \ ,$$

which was to be proved.

(21) If K = RA, both expressions $S_1(A,R,K)$ and $S_3(A,R,K)$ are zero, so assume that K < RA. Then the statement that

$$S_3(A,R,K) \leq S_1(A,R,K)$$

is equivalent to

$$\left(1 - \frac{K}{RA}\right)^R \geq 1 - \frac{K}{A} \ . \tag{19}$$

The assumptions of the theorem imply that the LHS of (19) is always nonnegative. If $K/A \geq 1$, then the RHS of (19) is nonpositive and (19) (and the theorem) follow forthwith. Note that expression (19) is the same as expression (7), which appeared in the proof of Theorem 14 in Section D.3, above. By the methods of the proof of Theorem 14, expression (19), and thus Theorem 21, are also true if K/A < 1.

E. AN EXCURSION--A SIMPLE "NONDETERMINISTIC" EXPERIMENT

The experiments of Section C all involve choosing exactly K circles from an array of circles. As an excursion, this section examines an experiment that is in some sense the simplest case in which a nondeterministic number of circles is chosen from the array. Some correspondences between this experiment and the results of Sections B and C are indicated. Other such "nondeterministic" experiments could be developed and explored, but that is not the focus of this paper.

Recall the array of circles shown in Figure 1; consider a general such array that is R rows by A columns (R and A are positive integers). Let K be some positive integer less than or equal to RA, define $p = K/(RA)$, and consider the following experiment:

> Color each circle in the array black with probability p, treating different circles independently of one another.

The number of black circles in the array is then a binomially distributed random variable (with parameters RA and p); the expected number of black circles is K.

A column in the array contains at least one black circle with probability $1-(1-p)^R$, thus the expected number of columns containing at least one black circle is

$$A\,[1-(1-p)^R],$$

which is the same as the formula given in Section B for the expected number of aircraft killed (in the combat process considered there). Substituting K/(RA) for p yields the formula $N_3(A,R,K)$ of Theorem 3 (Section C.1).

The expected number of white circles not located (vertically) under some black circle can also be derived. For $n=0,\ldots,R-1$, any particular column will have exactly n white circles at the top with probability $(1-p)^n p$, and all R circles of a column will be white with probability $(1-p)^R$. The expected number of white circles at the top of a column can then be evaluated as described in the proof of the Proposition of Section B. The overall number of white circles not located under some black circle is A times the expected number of white circles at the top of any particular column, or

$$\frac{A(1-p)}{p}\left[1-(1-p)^R\right],$$

which is the same as expression (5) in the Proposition of Section B. Substituting K/(RA) for p yields the formula $S_3(A,R,K)$ of Theorem 9 (Section C.2).

The correspondence between the experiment just described and the combat process described in Section B is clear, and thus it is not surprising that the indicated formulas are the same. The equality of the formulas for the expected number of columns containing at least one black circle and the expected number of white circles not located under some black circle with the formulas $N_3(A,R,K)$ and $S_3(A,R,K)$ is more interesting to note, especially because $N_3(A,R,K)$ has been proved to lie in the middle of the range of formulas $N_i(A,R,K)$ (see Section C.3) and a similar result might well hold for $S_3(A,R,K)$. (That is, the "nondeterministic" experiment produces results that are "not at the extremes" of the "deterministic" experiments.)

This equality of formulas may be essentially coincidental, however. The experiment described in Theorem 3 and the experiment just described are somewhat different. Not only does the former experiment choose exactly K circles from the array, it assumes that K/R is an integer. And even if K/R is an integer, the equality of formulas is an equality of expected values only; higher order moments of the number of columns with at least one black circle and the number of white circles not located under some black circle are not necessarily the same for the two experiments.

F. CONCLUSIONS

All of the formulas developed in Section C in the context of abstract probability problems represent possible formulas for computing a number of aircraft killed from a number of sorties killed. The spectrum of inequalities between these formulas, as given by Theorems 13 through 19 of Section C, provides potentially useful information for the modeling of sorties and attrition. In particular, the commonsense formula $N_1(A,R,K) = K/R$, which is the same as equation (3) of Section A and which has been used in a number of models, has been shown to be lower than several other reasonable formulas. In some cases another formula may be more appropriate. For example, Section B has related the formula $N_3(A,R,K)$ (which is also equation (4) of Section A) to a specific combat process.

It should be recalled, however, that even though K/R is low in the sense of Theorems 13 through 19, K itself, as determined by an attrition equation such as equation (2) of Section A, may overestimate the actual number of sorties killed (possibility for the reasons explained in Section A). One way of avoiding this problem is to adapt the "shoot-then-shoot-back" scheme (which is frequently used in the NAVMOD model [4] to compute numbers of sorties killed[1]) to make use of the formulas developed in Section C.2 for the expected numbers of successful sorties. If when a side suffers attrition, only that side's "successful sorties" (rather than the full number of surviving--i.e., "non-killed"--sorties) are allowed to shoot back, the overall number of sorties killed, K, may be lower than in the regular "shoot-then-shoot-back" procedure. In this case, the full range of formulas in Theorems 1 through 6 can be considered as reasonable candidates for converting sorties

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1 See Reference [4], Chapter IV, Sections A.1.c, A.2, and B.4 for a description of the "shoot-then-shoot-back" attrition procedure. References [5], [6], and [7] provide additional details concerning this procedure.

killed to aircraft killed; the dynamics of the particular combat situation being modeled may suggest that certain of these formulas are especially appropriate.

One problem to be explored further is to prove (or give counterexamples for) the inequalities between the $S_i(A,R,K)$ that were mentioned at the end of Section C. As stated earlier, these formulas in some sense correspond to possible ways of computing the number of successful sorties flown.

A more general problem is to compare the effects on a combat simulation of: 1) employing formulas such as those developed in this paper, as opposed to; 2) the straightforward method of assessing attrition more frequently, i.e., reducing the length of the time period so that sortie rates per time period never exceed 1.0. Are there circumstances in which one of these methods is to be preferred over the other?

Finally, some additional issues concerning the computation of the number of sorties killed, K or $\dot{T}_S$, should be addressed. For example, note that formulas (1) and (2) of Section A use the same effectiveness parameters and the same functional form. If it is assumed that formula (1) is reasonable when sortie rates are (less than or) equal to unity, does this necessarily imply that (2) is reasonable if sortie rates exceed unity? If not, what are some reasonable alternatives for computing $\dot{T}_S$?

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